Rigorous Analysis of Idealised Pathfinding Ants in Higher-Order Logic

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Pathfinding ants in abstract

Emergent behaviour

Q: *How can simple ants collectively solve the shortest path problem?*

A: *Instinct, stigmergy & auto-catalysis* $\Rightarrow$ Emergent behaviour

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Intriguing applications:

- Ant colony optimisation: TSP, Network routing, Vehicle routing
- Swarm intelligence: Robotics, Distributed search algorithms
- Efficient decentralised communication: Resource management, Blockchain systems
- Metaheuristics: Scheduling and Constraint satisfaction
- Emergent properties of AI: Group dynamics in virtual environments, Adaptive knowledge systems

Smell, move, deposit, repeat

In exploring the environment, each ant follow a simple, instinctive behaviour, without hidden complex reasoning.

Communicate

Communication among ants is ruled by that instinct, leading to an indirect form of interaction based on the local modification of the environment

⇒ Pheromone release and sensing guide the exploration of each individual, on the basis of the information marked by previous explorers on the environment.

Positive feedback

The areas collecting higher pheromone levels attract more and more ants, leading all ants to converging to the optimal path between two points.

A collective solution

The behaviour of the whole colony solve complex tasks like foraging, cooperative transport and nest maintenance,...

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"Karl Marx was right. Socialism works. It is just that he had the wrong species in mind."

– E.O. Wilson

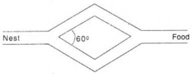
- Two branches connecting the colony nest to a food source
- Variable ratio $r := \frac{l}{s}$, where l is the length of the longer branch and s is the length of the shorter branch

Double bridge experiment

Conceptual setup

- Two branches connecting the colony nest to a food source
- Variable ratio $r := \frac{l}{s}$, where l is the length of the longer branch and s is the length of the shorter branch

Case 1: $r = 1$



- ○ At time t_0, no pheromone on either branch
- ⇒ Ants randomly choose a path, with small fluctuations *randomly* leading more ants onto one branch
- ○ By instinctive pheromone release while travelling, a *random initial imbalance* happens between the branches
- ⇒ By auto-catalysis, more and more ants choose the branch with higher pheromone (reinforcing loop), until all ants converge on that branch

Double bridge experiment

Conceptual setup

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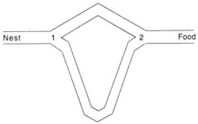
Case 2: $r > 1$



- ◦ At time t_0, ants still choose branches *randomly*
- ◦ Ants choosing the shorter branch return to nest faster, depositing pheromones more quickly

⇒ By instinctive pheromone smell, the higher pheromone concentration on the shorter path biases future choices

⇒ By auto-catalysis, ants rapidly converge on the shorter branch without the influence of initial randomness.

Our goal

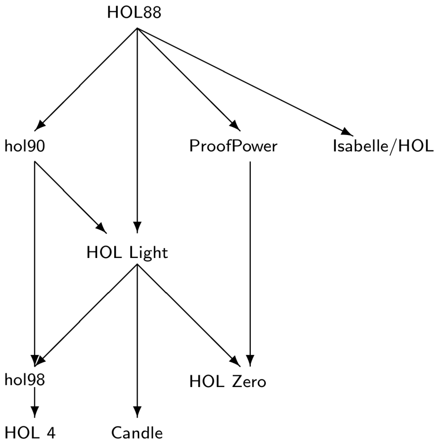
An application of computerised mathematics

1. *Mathematically* model the biological experiment,
2. *Functionally* simulate the colony dynamics, and
3. *Formally* verify the emergent behaviour of pathfinding

   by using the HOL Light proof assistant

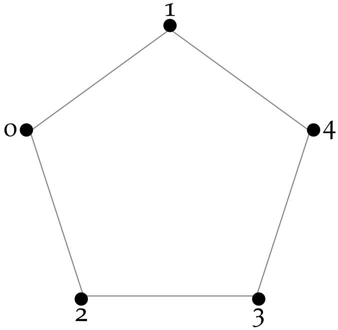
Brief glance at HOL Light

Basic facts



- Clean logical foundations ≈ *Principia Mathematica*
- LCF-style proof checker based on polymorphic simple type theory ≈ small class of *primitive inference rules* for creating theorems + *derived inference rules* to be programmed on top
  - ⇒ 10 primitive rules
  - ⇒ 2 conservative extension principles
  - ⇒ Axioms of choice, extensionality, and infinity
- Written as an OCaml program ≈ three datatypes for the logic: `hol_type`, `term`, and `thm`
- Goal-directed proof development ≈ tactic(al)s + automated methods (in the appropriate domains)

Despite its simple foundations, HOL Light includes a large library of mathematical results in topology, analysis, Euclidean geometry, QBF, floating point algorithms, FOL, limitative results, . . .



We define a new type `position` consisting exactly of the five nodes of the pentagon.

Pheromone levels on the intermediate nodes define the environmental information `STI` used by ants to communicate, which we encode as a 3d-vector of natural numbers to be updated during the evolution of the system.

An individual ant is just a pair of attributes $\langle \texttt{p}, \texttt{d} \rangle$:

- `p` is the node of the graph currently occupied by the ant (so it has type `position`)
- `d` is a boolean value denoting the direction the ant is moving on the graph (`T` for nest to food; `F` for the converse).

We thus convey to define the type $\texttt{ant} := \texttt{position} \times \texttt{bool}$.

To define the state of the colony at a discrete time t, we use the pair a vectors:

- an N-ary vector of items with type `ant`, where N is a parameter for the assumed size of the colony
- the 3d-vector of natural numbers denoting the current level of pheromones on the intermediate nodes.

To update the level of pheromones on the intermediate nodes of the graph, we compute the following equation

$$s'_p = s_p + \sum_{i=1}^{N} \delta_{\mathrm{pos}(a_i),p}$$

where s_p, s'_p are the levels of pheromones at the position p at two consecutive time steps, $\mathrm{pos}(a_i)$ is the position of the ant a_i, and $\delta_{x,y}$ (the Kroneker symbol) is 1 if $x = y$ and 0 otherwise.
The update of the vector `STI` is then defined in HOL Light as a functional program meeting the expected specification formalised in HOL (`NEW_STI`)

The “logic” of each individual ant is given by a function `NEW_ANT` (defined by case analysis) that taking a stigmergy vector and the attributes of an ant, update those attributes on the basis of the level of pheromones on the intermediate nodes.

Examples

- For `pos` = `P1` and `dir` = `T` (forward) we get:

  $$\texttt{NEW\_ANT sti (P1,T)} = \{\texttt{(P4,T)}\}$$

  meaning that the ant is forced to move in position `P4`, keeping the forward direction.
- For `pos` = `P4` and `dir` = `F` (backward), *if the levels of stigmergy* in `P1` and `P3` *are the same*, we have:

  $$\texttt{NEW\_ANT sti (P4,F)} = \{\texttt{(P1,F)},\texttt{(P3,F)}\}$$

  meaning that the ant has two possibilities: going on position `P1` or `P3` keeping, in any case, the backward direction.
  However, *if the two levels of stigmergy are not the same*, the ant chooses the node with higher pheromone concentration.

Informal insight

A colony evolves in any possible system (i.e., any potential distribution of the ants inhabiting the colony and correlated stigmergy) that complies with the individual specification of the foraging ants.

The function `NEW_SYSTEM` *compositionally* defines such a collective evolution by taking a system specification as input and returning a set of system specifications whose components provably meet the specifications defined by the functions `NEW_ANT` (for the individual components) and `NEW_STI` (for the environmental information) applied to the components of the input.

Simulations in HOL Light

From implicit to explicit specifications

1. Prove an alternative and equivalent procedural characterisation of the declarative definition of the function `NEW_SYSTEM` which the logical engine of HOL Light can evaluate as a purely functional expression;
2. Specialise that procedural specification for the case of a fixed number of ants;
3. Use this specialised version and the HOL Light conversion system to perform a certified computation of the simulation run and outcome.

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Example

Input: `{System (vector[(P1,T); (P2,F)])(vector[0; 0; 0])}`

Output after 30 iterations:

Ants status	Ph_1	Ph_2	Ph_3
$(P_1, F), (P_1, T)$	29	1	0
$(P_1, F), (P_0, F)$	28	2	1
$(P_4, T), (P_0, F)$	27	3	2
$(P_4, T), (P_1, F)$	25	4	3
$(P_1, T), (P_1, F)$	23	5	4
$(P_1, T), (P_4, T)$	22	6	5
$(P_0, F), (P_4, T)$	21	7	6
$(P_0, F), (P_1, T)$	19	8	7
$(P_1, F), (P_1, T)$	17	9	8
$(P_1, F), (P_0, F)$	16	10	9
$(P_4, T), (P_0, F)$	15	11	10
$(P_4, T), (P_1, F)$	13	12	11
$(P_3, T), (P_1, F)$	12	13	11

- By our full formalisation of the model, we are able to identify some invariant properties that are relevant for the emergence of the collective behaviour:
  - Stigmergy imbalance property: The pheromone level on the top of the pentagon is higher than the levels of the base nodes;
  - 2-step conservation principle: If the stigmergy values of a given system satisfy the imbalance property, any system evolving out of it does the same, and so does any system evolving out of the latter;
- Thus, we can prove an
  - Invariant lemma: The higher pheromone concentration on the shortest path is preserved by the evolution of any system of foraging ants satisfying the 2-step conservation principle for stigmergy.
- This suffices to formally prove our main

Theorem

Given an ant colony of any size, the convergence of the foraging ants on the shortest path always emerges after two evolution steps from any foraging ant system satisfying the stigmergy imbalance property and evolving in one step only into systems that preserve that property.

Put in perspective

Advantages of our methodology

- An alternative approach to modelling, simulating and verifying a (simple) collective adaptive system, based on logical tools and methods
- Use of a proof assistant guarantees the correctness of each step of the methodology, including simulation runs
- The emergence of the collective behaviour is proven as a mathematical theorem (or proven to be reducible to a condition, easier to check by traditional methods) about systems of any size.

Put in perspective

Future work

- Extension of the discrete model to more complex environment (e.g., 2d grids)
- Improvement of the functional encoding for simulation in terms of performance
- Combination of interactive theorem proving with automated reasoning and model checking for larger and realistic systems.

Many thanks for listening!