



Proofs, Programs, and Arrows for Epistemic Logics

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October 17, 2025

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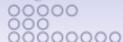
Categorical Semantics for Intuitionistic Belief

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24 Maggio 2019



Knowledge and Verification

*We adopt the view that an **an intuitionistic epistemic state (belief or knowledge) is the result of verification** where a verification is evidence considered sufficiently conclusive for practical purposes.*

[AP16]



A Two-Part Example from Mathematical Practice

Part I: Verification-based Belief

Your favourite top-rated mathematician has posted a proof sketch of a conjecture A in a very specialised mathematical field on her blog. Her informal argument is convincing, and her expertise justifies your trust in the conjecture. You are not an expert in that field, but she is; you do not have direct access to a detailed proof of A , but it is reasonable for you to believe that A does hold.

A Two-Part Example from Mathematical Practice

Part II: Verification-based Knowledge

Some time has passed since the mathematician's post on conjecture A . You are again surfing the Net looking for new results in the field A belongs, and come across another post about that conjecture. This time, a famous top-rated computer scientist's blog announces that A has been refuted. And there's more: Her refutation has been computerised, and the formal proof is freely available on the Net. If you know the programming language at the base of her computer program to refute A – a proof assistant, or a highly specialised theorem prover – you can read the proof and see that $\neg A$ holds, so that your original trust in A is misplaced.

In Plain Terms

We can read any epistemic formula $\Box A$ as asserting that A has a proof which is not necessarily specified in the process of verification, or more generally that it is verified that A holds in some not specified *constructive* sense.

This allows to apply verificationist/intuitionistic epistemic reasoning in various contexts which are not necessarily in the mathematical domain, such as:

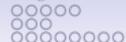
- Testimony of authority;
- Zero-knowledge protocols;
- Highly probable truth.

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- Zero-knowledge protocols;
- Highly probable truth.



Constructive Reasoning

Brouwer-Heyting-Kolmogorov Interpretation

According to the BHK semantics, a proposition A is true if there is a proof of it, and false if one can show that assuming A leads to a contradiction. More precisely:

- there is no proof of \perp ;
- a proof p of $A \wedge B$ consists of a pair $\langle a, b \rangle$ where a is a proof of A and b is a proof of B ;
- a proof p of $A \vee B$ is a pair $\langle n, q \rangle$ where $n = 0$ and q proves A , or $n = 1$ and q proves B ;
- a proof p of $A \rightarrow B$ is a rule which transforms any proof q of A into a proof $p(q)$ of B .



Constructive Epistemic Reasoning

Extending BHK

If we add an epistemic modal operator \Box to the language of intuitionistic propositional logic, we must extend the BHK interpretation to any formula $\Box A$.

The epistemic clause adopted is

- a proof p of $\Box A$ is a conclusive evidence of *verification* that A has a proof.

Constructive Epistemic Reasoning

Proofs and Verifications

1. Every proof is a verification;
2. That something is a proof is itself capable of proof.

Considering 1. and this extended semantics we have that

Intuitionistic Truth \Rightarrow Intuitionistic Knowledge/Belief

since, in general,

it is proved that A is verified

is a *weaker statement* than

it is proved that A

Constructive Epistemic Reasoning

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Intuitionistic Belief

Axiomatisation

IEL^-

Axioms

1. Axioms of propositional intuitionistic logic;
2. $\Box(A \rightarrow B) \rightarrow \Box A \rightarrow \Box B$; (K-scheme)
3. $A \rightarrow \Box A$. (co-reflection)

Rules

$$\frac{A \rightarrow B \quad A}{B} \text{MP}$$



Intuitionistic Knowledge

Axiomatisation

$\mathbb{I}EL$

Extend $\mathbb{I}EL^-$ by one of the following axiom schemas:

1. $\neg A \rightarrow \neg \Box A$;
2. $\neg(\Box A \wedge \neg A)$;
3. $\neg \Box \perp$;
4. $\neg\neg(\Box A \rightarrow A)$;
5. $\Box A \rightarrow \neg\neg A$.



Factivity of Knowledge

Classical vs Intuitionistic

Lemma

- (i) Over $\mathbb{I}EL^-$, the axiom schemas for intuitionistic knowledge are all equivalent.
- (ii) $\mathbb{I}EL^- \not\vdash \Box A \rightarrow A$.
- (iii) Over $\mathbb{I}K$, the axiom schemas for intuitionistic knowledge are organised according to the following hierarchy:

$$\begin{array}{c}
 \Box A \rightarrow A \\
 \Downarrow \\
 \neg(\Box A \wedge \neg A) \Leftrightarrow (\Box A \rightarrow \neg\neg A) \Leftrightarrow \neg\neg(\Box A \rightarrow A) \Leftrightarrow (\neg A \rightarrow \neg\Box A) \\
 \Downarrow \\
 \neg\Box \perp
 \end{array}$$

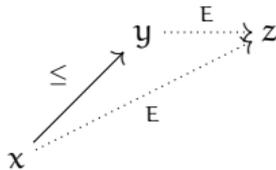


Relational Semantics

Sound and Complete

A model for $\mathbb{I}EL^-$ is a quadruple $\langle W, \leq, \nu, E \rangle$ where

- $\langle W, \leq, \nu \rangle$ is a standard model for intuitionistic propositional logic;
- E is a binary ‘knowledge’ relation on W such that:
 - if xEy , then $x \leq y$; and
 - if $x \leq y$ and yEz , then xEz ; graphically we have



- ν extends to a forcing relation \Vdash such that
 - $x \Vdash \Box A$ iff $y \Vdash A$ for all y such that xEy .

A model for $\mathbb{I}EL$ is a model for $\mathbb{I}EL^- \langle W, \leq, \nu, E \rangle$ where the relation E satisfies the seriality condition

- for any $x \in W$, there exists a $y \in W$ such that xEy .



NJ

Propositional Fragment

$$\frac{\Gamma}{\perp} \perp_J$$

$$\frac{}{\top} \top_J$$

$$\frac{\Gamma \quad \Delta}{A \wedge B} \wedge_I$$

$$\frac{\Gamma}{A \wedge B} \wedge_{\varepsilon_1}$$

$$\frac{\Gamma}{B} \wedge_{\varepsilon_2}$$

$$\frac{\Gamma}{A} \vee_{\mathcal{I}_1}$$

$$\frac{\Gamma}{B} \vee_{\mathcal{I}_2}$$

$$\frac{\Gamma \quad \Delta, [A]^1 \quad \Theta, [B]^2}{A \vee B \quad C} \vee_{\varepsilon:1,2}$$

$$\frac{\Gamma, [A]^1}{B} \rightarrow_{\mathcal{I}:1}$$

$$\frac{\Gamma \quad \Delta}{A \rightarrow B \quad A} \rightarrow_{\varepsilon}$$



Natural Deduction for Intuitionistic Belief

IEL⁻

Extend NJ_p by the following rule:

$$\frac{
 \begin{array}{ccc}
 \Gamma_1 & & \Gamma_n & & [A_1, \dots, A_n], \Delta \\
 \vdots & & \vdots & & \vdots \\
 \vdots & & \vdots & & \vdots \\
 \square A_1 & & \square A_n & & B
 \end{array}
 }{
 \square B
 }
 \square \mathcal{I}$$

where Γ and Δ are *multisets of formulas*, and A_1, \dots, A_n are *all* discharged.



Natural Deduction for Intuitionistic Knowledge

IEL

Extend NJ_p by the following *pair* of rules:

$$\frac{
 \begin{array}{c}
 \Gamma_1 \\
 \vdots \\
 \square A_1
 \end{array}
 \quad
 \begin{array}{c}
 \vdots \\
 \vdots \\
 \vdots
 \end{array}
 \quad
 \begin{array}{c}
 \Gamma_n \\
 \vdots \\
 \square A_n
 \end{array}
 \quad
 \begin{array}{c}
 [A_1, \dots, A_n], \Delta \\
 \vdots \\
 B
 \end{array}
 }{
 \square B
 }
 \quad \square \mathcal{I}
 \qquad
 \frac{
 \begin{array}{c}
 \Gamma \\
 \vdots \\
 \square A
 \end{array}
 \quad
 \begin{array}{c}
 [A], \Delta \\
 \vdots \\
 \perp
 \end{array}
 }{
 \perp
 }
 \quad \square \mathcal{E}$$

where: Γ and Δ are *finite multisets of formulas*; A_1, \dots, A_n are *all discharged* in $\square \mathcal{I}$; and A is discharged by $\square \mathcal{E}$.



Axioms and Rules

Equivalence of the Presentations

Lemma

$\Gamma \vdash_{\mathcal{L}} A$ *iff* $\Gamma \vdash_{\mathcal{L}} A$.



Axioms and Rules

Equivalence of the Presentations

Lemma

$\Gamma \vdash_{\mathbb{L}} A$ iff $\Gamma \vdash_{\mathbb{L}} A$.

Proof.

Assume $\Gamma \vdash_{\mathbb{L}} A$. We proceed by induction on the derivation. We only need to check that the axiom schemas for the modalities can be derived in the natural deduction calculus:

$[A \rightarrow B, A]$

- K:
$$\frac{\frac{\Box(A \rightarrow B) \quad \Box A}{\Box B} \quad \begin{array}{c} \vdots \\ B \end{array}}{\Box \mathcal{I}}$$
 ;

- co-reflection:
$$\frac{\frac{[A]^1}{\Box A} \Box \mathcal{I}}{A \rightarrow \Box A} \rightarrow \mathcal{I}:1$$
 ;

- schema $\Box A \rightarrow \neg\neg A$, we have in IEL
$$\frac{\frac{\frac{[\Box A]^3 \quad \frac{[A]^1 \quad [\neg A]^2}{\perp} \rightarrow \mathcal{E}}{\Box \mathcal{E}:1}}{\neg\neg A} \rightarrow \mathcal{I}:2}}{\Box A \rightarrow \neg\neg A} \rightarrow \mathcal{I}:3}$$



Axioms and Rules

Equivalence of the Presentations

Lemma

$\Gamma \vdash_{\mathbb{L}} A$ iff $\Gamma \vdash_{\mathbb{L}} A$.

Proof.

Conversely, assume $\Gamma \vdash_{\mathbb{L}} A$. We proceed by induction on the derivation. We only need to check that the natural deduction rules for the modalities are admissible in the axiomatic calculus, by leveraging the fact that \mathbb{L} is a normal modal logic.





Curry-Howard Correspondence

A Glossary

TYPE THEORY	LOGIC
type	proposition
term	proof
type constructor	logic connective
constructor	introduction rule
destructor	elimination rule
redex	proof detour
reduction	normalisation
normal form	normal proof
inhabitation	provability



Curry-Howard Correspondence

Proofs-as-Programs for NJ_p

$$\frac{\begin{array}{c} \Gamma, [A]^1 \\ \vdots \\ \vdots \\ \vdots \\ B \end{array}}{A \rightarrow B} \rightarrow\mathcal{I}:1$$

$$\frac{\begin{array}{c} \Gamma \\ \vdots \\ \vdots \\ \vdots \\ A \rightarrow B \end{array} \quad \begin{array}{c} \Delta \\ \vdots \\ \vdots \\ \vdots \\ A \end{array}}{B} \rightarrow\mathcal{E}$$

$$\frac{\Gamma \vdash t : B}{\Gamma - x : A \vdash \lambda x. t : A \rightarrow B} \rightarrow\text{-intro}$$

$$\frac{\Gamma \vdash t : A \rightarrow B \quad \Delta \vdash s : A}{\Gamma, \Delta \vdash ts : B} \rightarrow\text{-elim}$$



Curry-Howard Correspondence

Proofs-as-Programs for NJ_p

$$\frac{\begin{array}{c} \Gamma \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ A \end{array} \quad \begin{array}{c} \Delta \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ B \end{array}}{A \wedge B} \wedge \mathcal{I}$$

$$\frac{\begin{array}{c} \Gamma \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ A \wedge B \end{array}}{A} \wedge \mathcal{E}_1$$

$$\frac{\begin{array}{c} \Gamma \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ A \wedge B \end{array}}{B} \wedge \mathcal{E}_2$$

$$\frac{\Gamma_1 \vdash t_1 : A \quad \Gamma_2 \vdash t_2 : B}{\Gamma_1, \Gamma_2 \vdash (t_1, t_2) : A \times B} \times\text{-intro}$$

$$\frac{\Gamma \vdash t : A \times B}{\Gamma \vdash (\pi_1 t) : A} \times\text{-elim}_1$$

$$\frac{\Gamma \vdash t : A \times B}{\Gamma \vdash (\pi_2 t) : B} \times\text{-elim}_2$$



Curry-Howard Correspondence

Proofs-as-Programs for NJ_p

$$\frac{\begin{array}{c} \Gamma \\ \vdots \\ \vdots \\ \vdots \\ A \end{array}}{A \vee B} \vee\mathcal{I}_1$$

$$\frac{\begin{array}{c} \Gamma \\ \vdots \\ \vdots \\ \vdots \\ B \end{array}}{A \vee B} \vee\mathcal{I}_2$$

$$\frac{\begin{array}{c} \Gamma \\ \vdots \\ \vdots \\ \vdots \\ A \vee B \end{array} \quad \begin{array}{c} \Delta, [A]^1 \\ \vdots \\ \vdots \\ \vdots \\ C \end{array} \quad \begin{array}{c} \Theta, [B]^2 \\ \vdots \\ \vdots \\ \vdots \\ C \end{array}}{C} \vee\mathcal{E}:1,2$$

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash (\text{in}_1 t) : A + B} +\text{intro}_1$$

$$\frac{\Gamma \vdash s : B}{\Gamma \vdash (\text{in}_2 s) : A + B} +\text{intro}_2$$

$$\frac{\Gamma \vdash t : A + B \quad \Delta, x_1 : A \vdash t_1 : C \quad \Theta, x_2 : B \vdash t_2 : C}{\Gamma, \Delta, \Theta \vdash C(t, (x_1 : A.t_1), (x_2 : B.t_2)) : C} +\text{elim}:x_1, x_2$$



Curry-Howard Correspondence

Proofs-as-Programs for NJ_p

$$\frac{\begin{array}{c} \Gamma \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \perp \end{array}}{A} \perp_I$$

$$\frac{}{\top} \top_I$$

$$\frac{\Gamma \vdash t : \perp}{\Gamma \vdash (\text{ext}) : A} \text{ex falso}$$

$$\frac{}{\Gamma \vdash * : \top} \text{unit}$$



Curry-Howard Correspondence

Detours and Permutations in Proof-Terms

$$\frac{\frac{\begin{array}{c} \vdots \\ A_1 \end{array} \quad \begin{array}{c} \vdots \\ A_2 \end{array}}{A_1 \wedge A_2} \wedge \mathcal{I}}{A_i} \wedge \mathcal{E}_i$$

 $\sim \Xi$

$$\begin{array}{c} \vdots \\ A_i \end{array}$$

$$\frac{\frac{\begin{array}{c} \vdots \\ A_i \end{array}}{A_1 \vee A_2} \vee \mathcal{I}_i \quad \begin{array}{c} [A_1]^1 \\ \vdots \\ C \end{array} \quad \begin{array}{c} [A_2]^2 \\ \vdots \\ C \end{array}}{C} \vee \mathcal{E}:1,2$$

 $\sim \Xi$

$$\begin{array}{c} \vdots \\ A_i \\ \vdots \\ C \end{array}$$

$$\frac{\frac{\begin{array}{c} \vdots \\ B \end{array}}{A \rightarrow B} \rightarrow \mathcal{I} \quad \begin{array}{c} \vdots \\ A \end{array}}{B} \rightarrow \mathcal{E}$$

 $\sim \Xi$

$$\begin{array}{c} \vdots \\ B \end{array}$$



Curry-Howard Correspondence

Detours and Permutations in Proof-Terms

$$\begin{array}{c}
 \begin{array}{c} \vdots \\ A \vee B \end{array} \quad \begin{array}{c} [A]^1 \\ \vdots \\ C \end{array} \quad \begin{array}{c} [B]^2 \\ \vdots \\ C \end{array} \quad \begin{array}{c} \vdots \end{array} \\
 \hline
 C \quad \vee \mathcal{E}:1,2 \\
 \hline
 D \quad \mathcal{E}\text{-rule}
 \end{array}
 \quad \rightsquigarrow_{\exists}$$

$$\begin{array}{c}
 \begin{array}{c} \vdots \\ A \vee B \end{array} \quad \begin{array}{c} [A]^1 \\ \vdots \\ C \end{array} \quad \begin{array}{c} \vdots \\ D \end{array} \quad \begin{array}{c} [B]^2 \\ \vdots \\ C \end{array} \quad \begin{array}{c} \vdots \\ D \end{array} \\
 \hline
 \begin{array}{c} \vdots \\ A \vee B \end{array} \quad \begin{array}{c} \vdots \\ C \end{array} \quad \begin{array}{c} \vdots \\ D \end{array} \quad \begin{array}{c} \vdots \\ C \end{array} \quad \begin{array}{c} \vdots \\ D \end{array} \\
 \hline
 D \quad \vee \mathcal{E}:1,2
 \end{array}
 \quad \mathcal{E}\text{-rule}$$

$$\begin{array}{c}
 \begin{array}{c} \vdots \\ \perp \end{array} \quad \begin{array}{c} \vdots \\ A \end{array} \quad \perp \text{J} \\
 \hline
 B \quad \mathcal{E} / \text{ex falso-rule}
 \end{array}
 \quad \rightsquigarrow_{\exists} \quad \begin{array}{c} \vdots \\ \perp \\ B \end{array} \quad \perp \text{J}$$

Curry-Howard Correspondence

Normalisation

Lemma

Ξ -reductions have the Church-Rosser property.

Theorem (Weak Normalization)

For every $t : A$, there exists a unique Ξ -normal form.

Theorem (Strong Normalization)

For every $t : A$, all Ξ -reductions of $t : A$ terminate.



Curry-Howard Correspondence

Normalisation, Proof-Theoretically

Theorem (Strong normalisation, Prawitz 1965–71)

Given a deduction of A from Γ every reduction based on the conversions of Ξ terminates in a normal deduction.

Theorem (Subformula principle, Prawitz 1965–71)

Every formula occurring in a normal deduction is a subformula of the conclusion, or of some active assumption.



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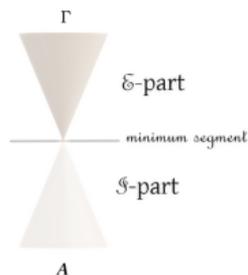
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Decidability, canonicity, consistency





Applicative Programming

IEL⁻ in Disguise

```

1 type ('a, 'e) validation_result =
2   | Success of 'a
3   | Failure of 'e list
4
5 type validation_error =
6   | UsernameTooShort
7   | InvalidEmail
8   | AgeTooYoung

```

(a) Validation result and error types

```

1 type user = {
2   username: string;
3   email: string;
4   age: int;
5 }

```

(b) User record definition

```

1 module ValidationApplicative = struct
2   type ('a, 'e) t = ('a, 'e) validation_result
3   let pure x = Success x
4   let apply f_result x_result =
5     match f_result, x_result with
6     | Success f, Success x -> Success (f x)
7     | Success _, Failure errs -> Failure errs
8     | Failure errs, Success _ -> Failure errs
9     | Failure errs1, Failure errs2 ->
10      Failure (errs1 @ errs2)
11   let (<*>) = apply
12 end

```

(c) Applicative instance

```

1 let validate_username username =
2   if String.length username >= 3
3   then Success username
4   else Failure [UsernameTooShort]
5
6 let validate_email email =
7   try
8     let at_pos = String.index email '@' in
9     let len = String.length email in
10    if at_pos > 0 && at_pos < len - 1 &&
11       not (String.contains_from email (at_pos + 1)
12           ~> '@')
13    then Success email
14    else Failure [InvalidEmail]
15  with Not_found -> Failure [InvalidEmail]
16
17 let validate_age age =
18   if age >= 13
19   then Success age
20   else Failure [AgeTooYoung]

```

(d) Individual validators

```

1 open ValidationApplicative
2
3 let create_user username email age =
4   { username; email; age }
5
6 let validate_user username email age =
7   pure create_user
8   <*> validate_username username
9   <*> validate_email email
10  <*> validate_age age

```



Modal Type Theory for Epistemic Logics

Constructor for Modal Terms

$$\frac{\Gamma_1 \vdash f_1 : \Box A_1 \quad \dots \quad \Gamma_n \vdash f_n : \Box A_n \quad x_1 : A_1, \dots, x_n : A_n, \Delta \vdash g : B}{\Gamma_1, \dots, \Gamma_n, \Delta \vdash (\text{box}[x_1, \dots, x_n]. g \text{ with } f_1, \dots, f_n) : \Box B}$$



Modal Type Theory for Epistemic Logics

Eliminator for Modal Terms

$$\frac{\Gamma \vdash f : \Box A \quad x : A, \Delta \vdash g : \perp}{\Gamma, \Delta \vdash (\text{unbox } f \text{ with } x.g) : \perp}$$



Detours for Belief

Basic Cases

$$\begin{array}{c}
 \Gamma \\
 \vdots \\
 \square \vec{A} \\
 \hline
 \square B \\
 \square \vec{D} \\
 \square F
 \end{array}
 \quad
 \begin{array}{c}
 [\vec{A}]^1, \vec{C} \\
 \vdots \\
 B \\
 \square x:1
 \end{array}
 \quad
 \begin{array}{c}
 \Delta \\
 \vdots \\
 \square \vec{D} \\
 \square F
 \end{array}
 \quad
 \begin{array}{c}
 [B, \vec{D}]^2, \vec{E} \\
 \vdots \\
 F \\
 \square x:2
 \end{array}
 \quad
 \rightsquigarrow_{\rho \square}
 \quad
 \begin{array}{c}
 \Gamma \\
 \vdots \\
 \square \vec{A} \\
 \square \vec{D} \\
 \square F
 \end{array}
 \quad
 \begin{array}{c}
 [\vec{A}]^1, \vec{C} \\
 \vdots \\
 B \\
 \square x:1
 \end{array}
 \quad
 \begin{array}{c}
 \Delta \\
 \vdots \\
 \square \vec{D} \\
 \square F
 \end{array}
 \quad
 \begin{array}{c}
 [\vec{D}]^1, \vec{E} \\
 \vdots \\
 F \\
 \square x:1
 \end{array}$$



Detours for Belief

Normalisation of Basic Cases

Lemma

Deductions in IEL^- strongly normalise w.r.t. the rewriting system obtained by adding ρ_{\square} to Ξ .

Detours for Belief

Normalisation of Basic Cases

Lemma

Deductions in IEL^- strongly normalise w.r.t. the rewriting system obtained by adding ρ_{\square} to Ξ .

Proof.

We define a CPS-translation $| - |$ from the λ -calculus of IEL^- -deductions to typed λ -calculus with products, sums, unit and empty types:

$$\begin{aligned} & \dots \\ & \dots \\ |\square A| & := (|A| \rightarrow q) \rightarrow q \end{aligned}$$

$$\begin{aligned} & \dots \\ & \dots \\ |\mathbf{box}[x_1, \dots, x_n]. g \text{ with } f_1, \dots, f_n| & := \lambda k. |f_1|(\lambda x_1. \dots |f_n|(\lambda x_n. k|g|) \dots) \end{aligned}$$

where q is an arbitrary atom.

Detours for Belief

Normalisation of Basic Cases

Lemma

Deductions in IEL^- strongly normalise w.r.t. the rewriting system obtained by adding ρ_{\square} to Ξ .

Proof.

$$\begin{array}{c}
 \Gamma \quad [A], \Delta \\
 \vdots \quad \vdots \\
 f \quad g \\
 \vdots \quad \vdots \\
 \square A \quad B \\
 \hline
 \square B \quad \square Z
 \end{array}
 \quad \mapsto \quad
 \begin{array}{c}
 |\Gamma| \\
 \vdots \\
 |f| \\
 \vdots \\
 (|A| \rightarrow q) \rightarrow q \\
 \hline
 q \\
 \hline
 (|B| \rightarrow q) \rightarrow q \quad \rightarrow_{\mathcal{I}:2}
 \end{array}
 \quad
 \begin{array}{c}
 [|A|]^1, |\Delta| \\
 \vdots \\
 |g| \\
 \vdots \\
 |B| \\
 \hline
 \rightarrow_{\mathcal{E}}
 \end{array}
 \quad
 \begin{array}{c}
 \frac{[|B| \rightarrow q]^2 \quad |B|}{q} \rightarrow_{\mathcal{E}} \\
 \frac{q \quad (|A| \rightarrow q)}{|A| \rightarrow q} \rightarrow_{\mathcal{I}:1} \\
 \hline
 q \rightarrow_{\mathcal{E}}
 \end{array}$$

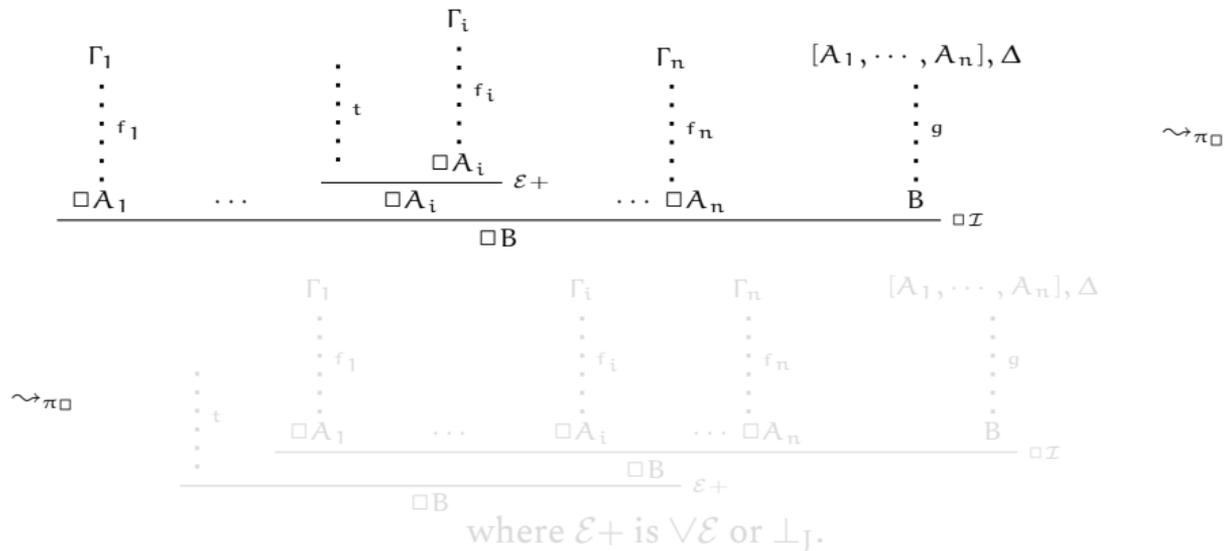
To prove strong normalisation it suffices now to prove that ρ_{\square} is preserved by this translation, which is almost straightforward.





Detours for Belief

Cases for Analyticity





Detours for Belief

Normalisation of the Extended System

Lemma

Deductions in IEL^- strongly normalise w.r.t. the rewriting system $\Xi + \rho_{\square} + \pi_{\square}$.



Detours for Belief

Normalisation of the Extended System

Lemma

Deductions in IEL^- strongly normalise w.r.t. the rewriting system $\Xi + \rho_{\Box} + \pi_{\Box}$.

Proof.

We define a new translation $\langle - \rangle$ from our modal λ -calculus to simple type theory with products, sums, unit and empty types as follows:

$$\begin{aligned} & \dots \\ & \dots \\ \langle \Box A \rangle & := \langle A \rangle \vee q \end{aligned}$$

\dots

\dots

$$\begin{aligned} & \langle \text{box}[x_1, \dots, x_n]. g \text{ with } f_1, \dots, f_n \rangle := \\ & := C(\langle f_n \rangle, x_n \cdot \dots \cdot C(\langle f_2 \rangle, x_2 \cdot C(\langle f_1 \rangle, x_1 \cdot \text{in}_1(\langle g \rangle), y_1 \cdot \text{in}_2(y_1)), y_2 \cdot \text{in}_2(y_2)) \cdot \dots, y_n \cdot \text{in}_2(y_n)) \end{aligned}$$

where q is an arbitrary atom.

Detours for Belief

Normalisation of the Extended System

Lemma

Deductions in IEL^- strongly normalise w.r.t. the rewriting system $\Xi + \rho_{\square} + \pi_{\square}$.

Proof.

$$\begin{array}{c}
 \Gamma_1 \quad \Gamma_2 \quad [A_1, A_2], \Delta \\
 \vdots \quad \vdots \quad \vdots \\
 \dots f_1 \quad \dots f_2 \quad \dots g \\
 \vdots \quad \vdots \quad \vdots \\
 \square A_1 \quad \square A_2 \quad B \quad \square \mathcal{I} \\
 \hline
 \square B \quad \square \mathcal{I}
 \end{array}
 \xrightarrow{\langle \rangle}
 \begin{array}{c}
 \langle \Gamma_1 \rangle \\
 \vdots \\
 \langle g \rangle \\
 \vdots \\
 \langle B \rangle \\
 \hline
 \langle B \rangle \vee q \quad \vee \mathcal{I}_1
 \end{array}
 \xrightarrow{\langle \rangle}
 \begin{array}{c}
 \langle \Gamma_2 \rangle \\
 \vdots \\
 \langle f_2 \rangle \\
 \vdots \\
 \langle A_2 \rangle \vee q
 \end{array}
 \xrightarrow{\langle \rangle}
 \begin{array}{c}
 \langle \Gamma_1 \rangle \\
 \vdots \\
 \langle f_1 \rangle \\
 \vdots \\
 \langle A_1 \rangle \vee q
 \end{array}
 \xrightarrow{\langle \rangle}
 \begin{array}{c}
 \langle \Gamma_1 \rangle, \langle \Gamma_2 \rangle, \langle A_1 \rangle^1, \langle A_2 \rangle^2, \langle \Delta \rangle \\
 \vdots \\
 \langle g \rangle \\
 \vdots \\
 \langle B \rangle \\
 \hline
 \langle B \rangle \vee q \quad \vee \mathcal{I}_1
 \end{array}
 \xrightarrow{\langle \rangle}
 \begin{array}{c}
 [q]^1 \\
 \hline
 \langle B \rangle \vee q \quad \vee \mathcal{I}_2
 \end{array}
 \xrightarrow{\langle \rangle}
 \begin{array}{c}
 [q]^2 \\
 \hline
 \langle B \rangle \vee q \quad \vee \mathcal{I}_2
 \end{array}
 \xrightarrow{\langle \rangle}
 \begin{array}{c}
 \langle B \rangle \vee q \quad \vee \mathcal{E}:1 \\
 \hline
 \langle B \rangle \vee q \quad \vee \mathcal{E}:2
 \end{array}
 \xrightarrow{\langle \rangle}
 \begin{array}{c}
 \langle B \rangle \vee q \quad \vee \mathcal{E}:2 \\
 \hline
 \langle B \rangle \vee q \quad \vee \mathcal{E}:2
 \end{array}$$

It is now straightforward to check that both ρ_{\square} and π_{\square} are preserved by this translation. □



Detours for Belief

Normalisation of the Extended System

Lemma

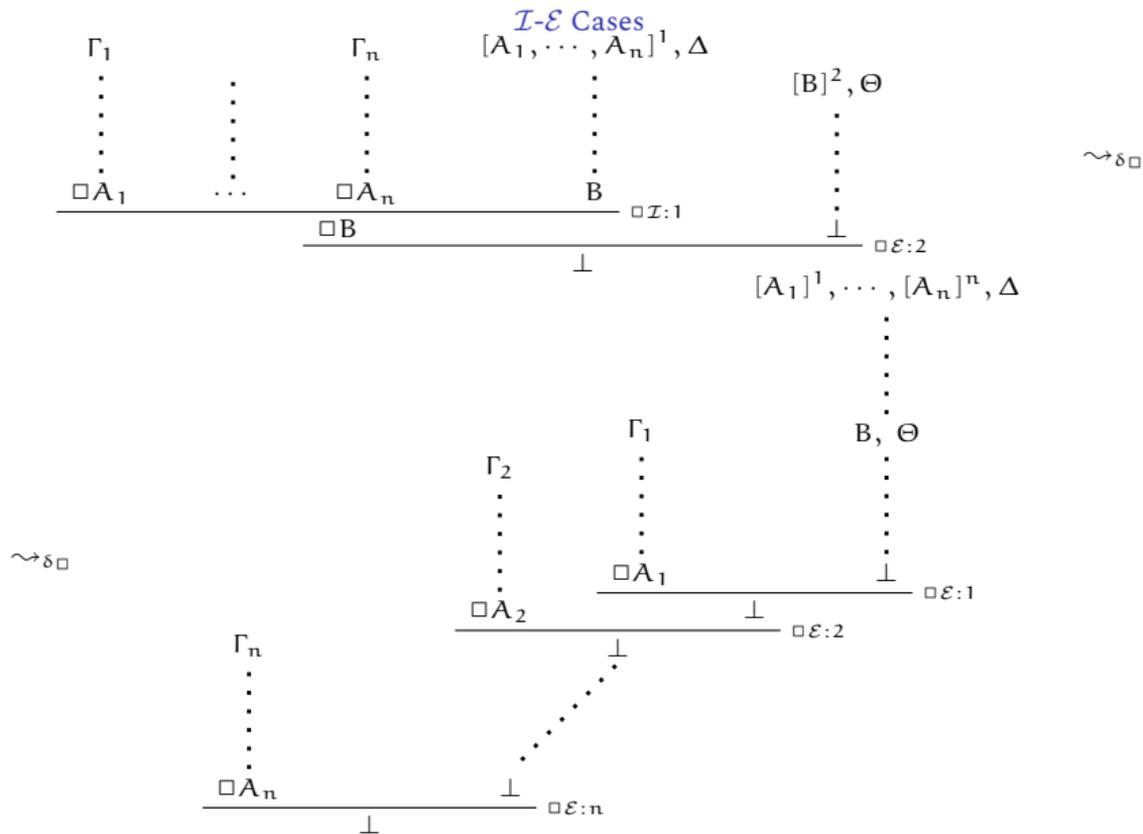
Deductions in IEL^- strongly normalise w.r.t. the rewriting system $\Xi + \rho_{\square} + \pi_{\square}$.

Corollary

Every IEL^- -deduction uniquely reduces to a normal IEL^- -deduction w.r.t. $\Xi + \rho_{\square} + \pi_{\square}$.



Detours for Knowledge





Detours for Knowledge

\mathcal{E} - \mathcal{I} Cases

$$\begin{array}{c}
 \Gamma \\
 \vdots \\
 \vdots \\
 \vdots \\
 \square \perp \\
 \hline
 \dots \quad \square \perp \quad [\perp] \quad \square \mathcal{E} \\
 \square \perp \quad \square \mathcal{I} \\
 \hline
 \square \perp
 \end{array}
 \rightsquigarrow \gamma \square \quad \dots \quad
 \begin{array}{c}
 \Gamma \\
 \vdots \\
 \vdots \\
 \vdots \\
 \vdots \\
 \square \perp
 \end{array}$$



Detours for Knowledge

\mathcal{E} - \mathcal{I} Cases

$$\begin{array}{c}
 \Gamma \\
 \vdots \\
 \vdots \\
 \vdots \\
 \square \perp \\
 \hline
 \dots \quad \square \perp \quad [\perp] \quad \square \varepsilon \\
 \square \perp \quad \square \varepsilon \\
 \hline
 \square \perp
 \end{array}
 \quad \rightsquigarrow \gamma \square \quad \dots \quad
 \begin{array}{c}
 \Gamma \\
 \vdots \\
 \vdots \\
 \vdots \\
 \vdots \\
 \square \perp
 \end{array}$$

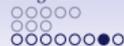


Detours for Knowledge

Normalisation of the Full System

Theorem

Deductions in IEL strongly normalise w.r.t. the rewriting system $\Xi + \rho_{\square} + \pi_{\square} + \delta_{\square} + \gamma_{\square}$.



Detours for Knowledge

Normalisation of the Full System

Theorem

Deductions in IEL strongly normalise w.r.t. the rewriting system $\Xi + \rho_{\square} + \pi_{\square} + \delta_{\square} + \gamma_{\square}$.

Proof.

Tweak the translation $\langle - \rangle$ from our modal λ -calculus for IEL^- to simple type theory with products, sums, unit and empty types as follows:

...

...

$$\langle \square A \rangle := \langle A \rangle \vee \perp$$

...

...

$$\langle \text{box}[x_1, \dots, x_n]. g \text{ with } f_1, \dots, f_n \rangle :=$$

$$:= C(\langle f_n \rangle, x_n. \dots C(\langle f_2 \rangle, x_2. C(\langle f_1 \rangle, x_1. \text{in}_1(\langle g \rangle), y_1. \text{in}_2(y_1)), y_2. \text{in}_2(y_2)) \dots, y_n. \text{in}_2(y_n))$$

$$\langle \text{unbox } f \text{ with } x.g \rangle := C(\langle f \rangle, x. \langle g \rangle, y.y).$$

Detours for Knowledge

Normalisation of the Full System

Theorem

Deductions in IEL strongly normalise w.r.t. the rewriting system $\Xi + \rho_{\square} + \pi_{\square} + \delta_{\square} + \gamma_{\square}$.

Proof.

$$\begin{array}{c}
 \Gamma \quad [A], \Delta \\
 \vdots \quad \vdots \\
 \dots f \quad \dots g \\
 \vdots \quad \vdots \\
 \square A \quad \perp \\
 \hline
 \perp \quad \square \varepsilon
 \end{array}
 \xrightarrow{\langle \rangle}
 \begin{array}{c}
 \langle \Gamma \rangle \quad \langle [A] \rangle^1, \langle \Delta \rangle \\
 \vdots \quad \vdots \\
 \dots \langle f \rangle \quad \dots \langle g \rangle \\
 \vdots \quad \vdots \\
 \langle A \rangle \vee \perp \quad \perp \\
 \hline
 \perp \quad [\perp]^1 \quad \vee \varepsilon : 1
 \end{array}$$

This translation preserves δ_{\square} and γ_{\square} too, by means of applications of rewritings in Ξ concerning \vee -detours and permutations.



Analyticity

Subformula Property and Corollaries

Theorem (Subformula Property)

Every formula B occurring in a normal L -deduction f of A from assumptions Γ is a subformula of A or of some formula in Γ .

Corollary

1. *Consistency: The logics of intuitionistic belief and intuitionistic knowledge are consistent.*
2. *Decidability: The logics of intuitionistic belief and intuitionistic knowledge are decidable.*
3. *Canonicity: In any normal L -deduction of A , the last rule applied is the introduction rule for the main connective of A .*
4. *Admissibility of Reflection: If $L \vdash \Box A$, then $L \vdash A$.*
5. *Disjunction Property: If $L \vdash A \vee B$, then $L \vdash A$ or $L \vdash B$.*
6. *\Box -Primality: If $L \vdash \Box A \vee \Box B$, then $L \vdash A$ or $L \vdash B$.*
7. *Modal Disjunction Property: If $L \vdash \Box(A \vee B)$, then $L \vdash \Box A$ or $L \vdash \Box B$.*
8. *Proper Inclusion: $IEL^- \subsetneq IEL$.*

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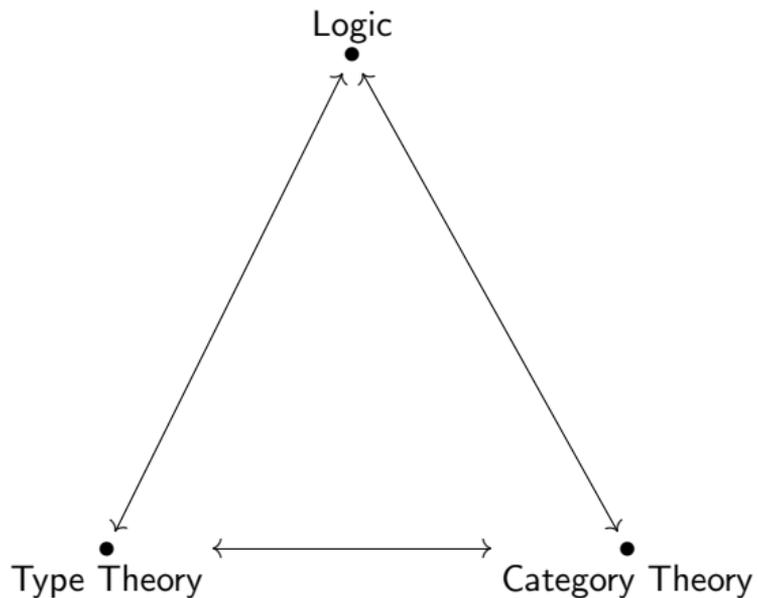
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Curry-Howard-Lambek Correspondence

Computational Trinitarianism





Curry-Howard-Lambek Correspondence

A Glossary

LOGIC	TYPE THEORY	CATEGORY THEORY
proposition	type	object
proof	term	arrow
theorem	inhabitant	element-arrow
conjunction	product type	product
true	unit type	terminal object
implication	function type	exponential
disjunction	sum type	(weak) coproduct
false	empty type	(weak) initial object

Category Theory

A Primer

Definition

A **category** \mathcal{C} consists of the following data:

- a collection of **objects** \mathcal{C}_0 , generally denoted \mathcal{C} also;
- a collection of **arrows** \mathcal{C}_1 ;
- to each arrow $f : \mathcal{C}_1$, a pair of objects $\langle \text{source}(f), \text{target}(f) \rangle$ is associated. We write $f : X \rightarrow Y$. The collection of arrows from X to Y is denoted by $\text{hom}(X, Y)$;
- to each object $X : \mathcal{C}_0$, an arrow $\text{id}_X : X \rightarrow X$ is associated.
- to each pair of arrows f, g such that $\text{source}(g) = \text{target}(f)$ is associated an arrow $g \circ f : \text{source}(f) \rightarrow \text{target}(g)$.

These data need to satisfy specific conditions:

Associativity: For any composable arrows f, g and g, h , we have $h \circ (g \circ f) = (h \circ g) \circ f$;

Identity: For any $f : X \rightarrow Y$, we have $\text{id}_Y \circ f = f = f \circ \text{id}_X$.

Category Theory

A Primer

Definition

Let \mathcal{C} and \mathcal{D} be categories. A **functor** $\mathfrak{F} : \mathcal{C} \rightarrow \mathcal{D}$ consists of the following data:

- A function $\mathfrak{F}_0 : \mathcal{C} \rightarrow \mathcal{D}$, denoted \mathfrak{F} also.
- For each $X, Y : \mathcal{C}$, a function $\mathfrak{F}_{X,Y} : \text{hom}_{\mathcal{C}}(X, Y) \rightarrow \text{hom}_{\mathcal{D}}(\mathfrak{F}X, \mathfrak{F}Y)$, denoted \mathfrak{F} also;

satisfying the following conditions:

- (i) For each $X : \mathcal{C}$, we have $\mathfrak{F}(\text{id}_X) = 1_{\mathfrak{F}(X)}$; and
- (ii) For each $X, Y, Z : \mathcal{C}$, and $f : \text{hom}_{\mathcal{C}}(Y, Z)$ and $g : \text{hom}_{\mathcal{C}}(X, Y)$, we have $\mathfrak{F}(g \circ f) = \mathfrak{F}g \circ \mathfrak{F}f$.

Category Theory

A Primer

Definition

For functors $\mathfrak{F}, \mathfrak{G} : \mathcal{C} \rightarrow \mathcal{D}$, a **natural transformation** $\gamma : \mathfrak{F} \Rightarrow \mathfrak{G}$ consists of the following data:

- For each $X : \mathcal{C}$, an arrow $\gamma_X : \text{hom}_{\mathcal{D}}(\mathfrak{F}X, \mathfrak{G}X)$, giving the **components** of γ ;
satisfying the **naturality** condition

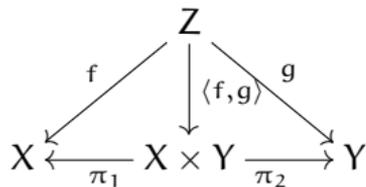
For each $X, Y : \mathcal{C}$ and $f : \text{hom}_{\mathcal{C}}(X, Y)$, we have $\mathfrak{G}f \circ \gamma_X = \gamma_Y \circ \mathfrak{F}f$.



Category Theory

A Primer

- **Product of objects** X, Y : Object $X \times Y$ of \mathcal{C} together with projection arrows $\pi_1 : X \times Y \rightarrow X$, $\pi_2 : X \times Y \rightarrow Y$ such that any diagram



commutes for a unique $\langle f, g \rangle : Z \rightarrow X \times Y$.

- **Terminal object**: Object 1 of \mathcal{C} such that for any other $X : \mathcal{C}$, there exists a unique arrow $!_X : X \rightarrow 1$.



Category Theory

A Primer

- **Exponential of objects Y and X :** An object Y^X of \mathcal{C} together with an arrow $eval : X \times Y^X \rightarrow Y$ such that any diagram

$$\begin{array}{ccc}
 X \times Z & & \\
 \text{id}_X \times \lambda_f \downarrow & \searrow f & \\
 X \times Y^X & \xrightarrow{\quad eval \quad} & Y
 \end{array}$$

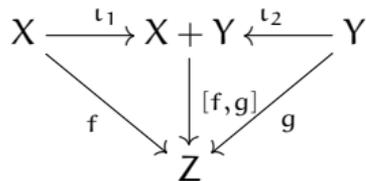
commutes for a unique $\lambda_f : Z \rightarrow Y^X$.



Category Theory

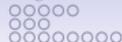
A Primer

- **Coproduct of objects** X, Y : Object $X + Y$ of \mathcal{C} together with injection arrows $\iota_1 : X \rightarrow X + Y$, $\iota_2 : Y \rightarrow X + Y$ such that any diagram



commutes for a unique $[f, g] : X + Y \rightarrow Z$.

- **Initial object**: Object 0 of \mathcal{C} such that for any other $X : \mathcal{C}$, there exists a unique arrow $0_X : 0 \rightarrow X$.



Cartesian Closed Categories

Proof-Theoretic Semantics for \vee -free NJ_p

Cartesian closed categories (CCCs) provide the semantics *and* proof theory of disjunction-free propositional intuitionistic logic:

- Objects correspond to formulae, and an arrow from A to B corresponds to a deduction $A \vdash_{\text{NJ}} B$ between the corresponding formulae;
- Conjunction is modelled by products, by using the adjunction

$$C \vdash_{\text{NJ}_p} A \wedge B \quad \simeq \quad C \vdash_{\text{NJ}_p} A \ \& \ C \vdash_{\text{NJ}_p} B;$$

- Implication is modelled by exponentials, by using the adjunction

$$A \wedge B \vdash_{\text{NJ}_p} C \quad \simeq \quad A \vdash_{\text{NJ}_p} B \rightarrow C;$$

- Top is modelled by terminal object 1 .

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Bi-Cartesian Closed Categories

Proof-Theoretic Semantics for NJ_p

If a CCC has finite (weak) coproducts it is called bi-Cartesian closed.

Bi-CCCs provide the semantics *and* proof theory of propositional intuitionistic logic:

- Disjunction is modelled by (weak) coproducts, by using the adjunction

$$A \vee B \vdash_{\text{NJ}} C \quad \simeq \quad A \vdash_{\text{NJ}} C \& B \vdash_{\text{NJ}} C;$$

- Bottom is modelled by (weak) initial object 0.

Bi-Cartesian Closed Categories

Proof-Theoretic Semantics for NJ_p

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Categories for Epistemic States

Monoidal Endofunctors

Given a CCCat \mathcal{C} , a monoidal endofunctor consists of a functor $\mathfrak{F} : \mathcal{C} \rightarrow \mathcal{C}$ together with

- a natural transformation

$$m_{A,B} : \mathfrak{F}A \times \mathfrak{F}B \rightarrow \mathfrak{F}(A \times B);$$

- a morphism

$$m_1 : 1 \rightarrow \mathfrak{F}1,$$

preserving the monoidal structure of \mathcal{C} . These are called structure morphisms of \mathfrak{F} .



Categories for Epistemic States

Pointed Endofunctors

Given any category \mathcal{C} , an endofunctor $\mathfrak{F} : \mathcal{C} \rightarrow \mathcal{C}$ is pointed iff there exists a natural transformation

$$\pi : \text{Id}_{\mathcal{C}} \Rightarrow \mathfrak{F}$$

$$\pi_A : A \rightarrow \mathfrak{F}A$$

$$\begin{array}{ccc}
 A & \xrightarrow{\pi_A} & \mathfrak{F}A \\
 \downarrow f & & \downarrow \mathfrak{F}f \\
 B & \xrightarrow{\pi_B} & \mathfrak{F}B
 \end{array}$$

π is called the point of \mathfrak{F} .

Categories for Epistemic States

Monoidal Natural Transformations

Given a monoidal category \mathcal{C} , and monoidal endofunctors $\mathfrak{F}, \mathfrak{G} : \mathcal{C} \rightarrow \mathcal{C}$, a natural transformation $\kappa : \mathfrak{F} \Rightarrow \mathfrak{G}$ is monoidal when the following commute:

$$\begin{array}{ccc} \mathfrak{F}A \times \mathfrak{F}B & \xrightarrow{m_{A,B}^{\mathfrak{F}}} & \mathfrak{F}(A \times B) \\ \kappa_A \times \kappa_B \downarrow & & \downarrow \kappa_{A \times B} \\ \mathfrak{G}A \times \mathfrak{G}B & \xrightarrow{m_{A,B}^{\mathfrak{G}}} & \mathfrak{G}(A \times B) \end{array}$$

and

$$\begin{array}{ccc} 1 & \xrightarrow{m_1^{\mathfrak{F}}} & \mathfrak{F}(1) \\ \parallel & & \downarrow \kappa_1 \\ 1 & \xrightarrow{m_1^{\mathfrak{G}}} & \mathfrak{G}(1) \end{array}$$



Categories for Epistemic States

Categorical Interpretation of our Calculi

Definition

An IEL^- -category is given by a bi- CCCat \mathcal{C} together with a monoidal pointed endofunctor \mathfrak{K} whose point κ is monoidal.

When \mathfrak{K} preserves the initial object 0 of \mathcal{C} up-to-isomorphism, we say that \mathfrak{K} is *dense*, and \mathcal{C} is an IEL -category.



Categories for Epistemic Proofs

\Box -Extensionality

$$\frac{\begin{array}{c} \Gamma \\ \vdots \\ \Box A \end{array} \quad [A] \quad \Box \mathcal{I}}{\Box A} \rightsquigarrow_{\eta_{\Box}} \begin{array}{c} \Gamma \\ \vdots \\ \Box A \end{array}$$

Lemma

\mathcal{L} -deductions strongly normalise w.r.t. the rewriting system $\Xi + \rho_{\Box} + \delta_{\Box} + \gamma_{\Box} + \eta_{\Box}$



Categories for Epistemic Proofs

\Box -Extensionality

$$\frac{\begin{array}{c} \Gamma \\ \vdots \\ \Box A \end{array} \quad [A] \quad \Box \mathcal{I}}{\Box A} \quad \rightsquigarrow_{\eta_{\Box}} \quad \begin{array}{c} \Gamma \\ \vdots \\ \Box A \end{array}$$

Lemma

L-deductions strongly normalise w.r.t. the rewriting system $\Xi + \rho_{\Box} + \delta_{\Box} + \gamma_{\Box} + \eta_{\Box}$



Categorical Interpretation

Soundness

Theorem

Let \mathcal{C} be an IEL^- -category. Then there is a canonical interpretation $\llbracket - \rrbracket$ of IEL^- in \mathcal{C} such that

- a formula A is mapped to a \mathcal{C} -object $\llbracket A \rrbracket$;
- a deduction f of $A_1, \dots, A_n \vdash_{\text{IEL}^-} B$ is mapped to an arrow

$$\llbracket f \rrbracket : \llbracket A_1 \rrbracket \times \dots \times \llbracket A_n \rrbracket \rightarrow \llbracket B \rrbracket;$$

- for any two deductions f and g which are equal modulo $\Xi + \rho_{\square} + \eta_{\square}$ -rewritings, we have $\llbracket f \rrbracket = \llbracket g \rrbracket$.

The analogous soundness result holds for any IEL -category, by considering the system of rewritings $\Xi + \rho_{\square} + \delta_{\square} + \gamma_{\square}$ extended by η_{\square} .



Categorical Interpretation

Soundness

Proof Sketch.

By structural induction on $f : \vec{A} \vdash_{\text{IEL}} B$.

The intuitionistic cases are interpreted according to the remarks about bi-CCCs.

Let $\langle \square, m, \kappa \rangle$ be the monoidal pointed endofunctor of \mathcal{C} , its structure morphisms, and its point.

The deduction
$$\frac{f_1 : \Gamma_1 \vdash \square A_1 \quad \cdots \quad f_n : \Gamma_n \vdash \square A_n \quad g : [A_1, \dots, A_n], C_1, \dots, C_m \vdash B}{\Gamma_1, \dots, \Gamma_n, C_1, \dots, C_m \vdash \square B}$$

is mapped to

$$(\square[g]) \circ m_{[A_1], \dots, [A_n], [C_1], \dots, [C_m]} \circ [f_1] \times \cdots \times [f_n] \times \kappa_{[C_1]} \times \cdots \times \kappa_{[C_m]}.$$

Identity modulo η_{\square} holds in the category \mathcal{C} by functoriality of \square .

Identity modulo ρ_{\square} is also valid by naturality of m and κ .

For IEL, we furthermore have the density property of the functor \square , which validates identity of deductions modulo δ_{\square} and γ_{\square} .



Categorical Interpretation

Soundness

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By structural induction on $f : \vec{A} \vdash_{\text{IEL}} B$.

The intuitionistic cases are interpreted according to the remarks about bi-CCCs.

Let $\langle \square, m, \kappa \rangle$ be the monoidal pointed endofunctor of \mathcal{C} , its structure morphisms, and its point.

The deduction
$$\frac{f_1 : \Gamma_1 \vdash \square A_1 \quad \cdots \quad f_n : \Gamma_n \vdash \square A_n \quad g : [A_1, \dots, A_n], C_1, \dots, C_m \vdash B}{\Gamma_1, \dots, \Gamma_n, C_1, \dots, C_m \vdash \square B}$$

is mapped to

$$(\square[g]) \circ m_{[A_1], \dots, [A_n], [C_1], \dots, [C_m]} \circ [f_1] \times \cdots \times [f_n] \times \kappa_{[C_1]} \times \cdots \times \kappa_{[C_m]}.$$

Identity modulo η_\square holds in the category \mathcal{C} by functoriality of \square .

Identity modulo ρ_\square is also valid by naturality of m and κ .

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Categorical Interpretation

Soundness

Proof Sketch.

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Categorical Interpretation

Completeness

Theorem

If the interpretation of two L-deductions is equal in all L-categories, then the two deductions are equal modulo $\Xi + \rho_{\square} + \delta_{\square} + \gamma_{\square} + \eta_{\square}$.

Proof Sketch.

We proceed by constructing a term model for the modal λ -calculus for L-deductions. Consider the following category \mathcal{M} :

- its objects are formulas;
- an arrow $f : A \rightarrow B$ is an L-deduction of B from A modulo rewritings in $\Xi + \rho_{\square} + \delta_{\square} + \gamma_{\square} + \eta_{\square}$;
- identities are given by assuming a hypothesis;
- composition is given by transitivity of deductions.

\mathcal{M} has the structure required to be an L-categories in virtue of the rewriting system imposed on deductions, so that if an equation between interpreted L-deductions holds in all L-categories, then it holds also in \mathcal{M} , so that those deductions are equal whenever they normalise to the same proof w.r.t. $\Xi + \rho_{\square} + \delta_{\square} + \gamma_{\square} + \eta_{\square}$ -rewritings. □

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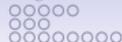
Conclusions

Recap

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- ▷ We established their proof-theoretic well-behaviour by proving key structural properties, including normalization theorems.
- ▷ We provided a categorical interpretation of epistemic modalities as functors, giving a precise proof-theoretic semantics to intuitionistic knowledge and belief.

Future Directions

- A comparative analysis and **inter-translatability results** with existing sequent calculi for these logics.
- **Modular extensions** of our base calculus (IEL^-) to model logics beyond epistemic reasoning.
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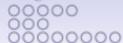
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Thank you for listening!





References I



Sergei Artemov and Tudor Protopopescu.

Intuitionistic epistemic logic.

The Review of Symbolic Logic, 9.2:266–298, 2016.



Guram Bezhanishvili and Wesley H. Holliday.

Locales, Nuclei, and Dragalin Frames.

In Lev D. Beklemishev, Stéphane Demri, and András Maté, editors, *Advances in Modal Logic 11, proceedings of the 11th conference on "Advances in Modal Logic," held in Budapest, Hungary, August 30 - September 2, 2016*, pages 177–196. College Publications, 2016.



Valeria de Paiva and Eike Ritter.

Basic constructive modality.

Logic without Frontiers: Festschrift for Walter Alexandre Carnielli on the occasion of his 60th Birthday, pages 411–428, 2011.



Melvin Fitting.

Modal proof theory.

In *Handbook of modal logic*, pages 85–138. Elsevier, 2007.

References II



Joachim Lambek and Philip J. Scott.

Introduction to higher-order categorical logic, volume 7.

Cambridge University Press, 1988.



Saunders MacLane.

Categories for the working mathematician, volume 5.

Springer Science & Business Media, 2013.



Paolo Mancosu, Sergio Galvan, and Richard Zach.

An Introduction to Proof Theory: Normalization, Cut-Elimination, and Consistency Proofs.

Oxford University Press, 2021.



Conor McBride and R.A. Paterson.

Applicative programming with effects.

Journal of functional programming, 18(1):1–13, 2008.



Cosimo Perini Brogi.

Curry–Howard–Lambek correspondence for intuitionistic belief.

Studia Logica, 109(6):1441–1461, 2021.

References III

-  **Cosimo Perini Brogi.**
From Applicative Programming to Verification-based Knowledge: A Curry-Howard-Lambek Reading.
In Proceedings of the 7th International Workshop on Artificial Intelligence and Formal Verification, Logic, Automata, and Synthesis, OVERLAY 2025, Bologna, Italy, October 26, 2025, To Appear.
-  **Dag Prawitz.**
Natural deduction: A proof-theoretical study.
Courier Dover Publications, 1965.
-  **Charles A. Stewart, Valeria de Paiva, and Natasha Alechina.**
Intuitionistic modal logic: A 15-year retrospective.
J. Log. Comput., 28(5):873–882, 2018.
-  **Morten H. Sørensen and Pawel Urzyczyn.**
Lectures on the Curry-Howard isomorphism, volume 149 of *Studies in Logic and the Foundations of Mathematics*.
Elsevier, 2006.